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Quantum Jordanian twist

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Abstract

The quantum deformation of the Jordanian twist $\mathcal{F}_{q\mathcal{J}}$ for the standard quantum Borel algebra $U_q(B)$ is constructed. It gives the family $U_{q\mathcal{J}}(B)$ of quantum algebras depending on parameters ξ and h. In a generic point these algebras represent the hybrid (standard–nonstandard) quantization. The quantum Jordanian twist can be applied to the standard quantization of any Kac–Moody algebra. The corresponding classical r-matrix is a linear combination of the Drinfeld–Jimbo and the Jordanian ones. The two-parametric families of Hopf algebras obtained here are smooth and for the limit values of the parameters the standard and nonstandard quantizations are recovered. The twisting element $\mathcal{F}_{q\mathcal{J}}$ also has correlated limits; in particular when q tends to unity it acquires the canonical form of the Jordanian twist. To illustrate the properties of the quantum Jordanian twist we construct the hybrid quantizations for $U(sl(2))$ and for the corresponding affine algebra $U(\widehat{sl(2)})$. The universal quantum $\mathcal{R}$-matrix and its defining representation are presented.

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1. Introduction

It has been known for a long time [1] that a Hopf algebra $\mathcal{A}(m, \Delta, \epsilon, S)$ with multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\epsilon: \mathcal{A} \to C$ and antipode $S : \mathcal{A} \to \mathcal{A}$ can be transformed with an invertible (twisting) element $\mathcal{F} \in \mathcal{A} \otimes \mathcal{A}$, $\mathcal{F} = \sum f_i^{(1)} \otimes f_i^{(2)}$, into a twisted one $\mathcal{A}_{\mathcal{F}}(m, \Delta_{\mathcal{F}}, \epsilon, S_{\mathcal{F}})$ that has the same multiplication and counit but different coproduct and antipode. The twisted coproduct is given by

$$\Delta_{\mathcal{F}}(a) = \mathcal{F}\Delta(a)\mathcal{F}^{-1}. \tag{1.1}$$

The twisting element has to satisfy the equations

$$(\epsilon \otimes \mathrm{i}d)(\mathcal{F}) = (\mathrm{i}d \otimes \epsilon)(\mathcal{F}) = 1 \tag{1.2}$$

$$\mathcal{F}_{12}(\Delta \otimes \mathrm{i}d)(\mathcal{F}) = \mathcal{F}_{23}(\mathrm{i}d \otimes \Delta)(\mathcal{F}). \tag{1.3}$$

There are several special types of twist. For our purposes the most interesting will be the factorizable twist whose twisting element satisfies the factorized twist equations [2]:

$$\begin{aligned}(\Delta \otimes \mathrm{i}d)(\mathcal{F}) &= \mathcal{F}_{13}\mathcal{F}_{23}\\ (\mathrm{i}d \otimes \Delta_{\mathcal{F}})(\mathcal{F}) &= \mathcal{F}_{12}\mathcal{F}_{13}.\end{aligned} \tag{1.4}$$

If the initial Hopf algebra $\mathcal{A}$ is quasitriangular with universal $\mathcal{R}$-matrix $\mathcal{R}$ then $\mathcal{A}_{\mathcal{F}}$ is the twisted Hopf algebra whose universal element $\mathcal{R}_{\mathcal{F}}$ is related to the initial one by

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\,\mathcal{R}\,\mathcal{F}^{-1}. \tag{1.5}$$

The Jordanian twist with the two-dimensional carrier subalgebra $B(2)$,

$$[H, E] = E$$

defined by the canonical twisting element

$$\mathcal{F}^{c}_{\mathcal{J}} = \mathrm{e}^{H\otimes\sigma} \qquad \sigma = \ln(E+1) \tag{1.6}$$

is the first and very important example [3] of a nontrivial twist with explicitly defined twisting element.

It was proved in [3] that there exist mixed quantizations combining the properties of the standard deformations and those of the twisted algebras. Up to now a considerable number of studies devoted to combined (standard–nonstandard) quantizations have been performed (especially for the case of $U(sl(2))$, $U(gl(2))$ and the corresponding quantum groups). (See, for example, the works by Gerstenhaber *et al* [4], Kupershmidt [5], Ballesteros *et al* [6], Abdesselam *et al* [7, 8], Aneva *et al* [9] and references therein. The last work contains a kind of review of the situation with the combined deformations and we shall return to it in section 4.) However, the question remains of whether it is possible to supply the combined quantization with a twisting element that would bring it 'back' to the standard quantum algebra (such as $U_q(sl(2))$). This 'going back' procedure is a limit process. In fact there are two limits to be considered: they are related to the behaviour of two main parameters: the deformation parameter $h = \ln q$ and the twisting parameter ξ. Recently a q-analogue $(\mathcal{F}^{c}_{\mathcal{J}})_q$ of the Jordanian twisting element (1.6) was constructed [10]. It transforms the standard quantization $U_q(sl(2))$ into the combined deformation $(U_q(sl(2)))_{\mathcal{J}}$ and the inverse operator $(\mathcal{F}^{c}_{\mathcal{J}})^{-1}_q$ obviously brings the algebra $(U_q(sl(2)))_{\mathcal{J}}$ back into the standard deformation. For us it is important to notice that the Hopf algebra $(U_q(sl(2)))_{\mathcal{J}}$ has no classical limit for $h \to 0$.

In this paper we demonstrate that there are other sheets of combined quantizations for which both limits exist: for $h \to 0$ (the standard q-deformation) and for $\xi \to 0$ (the nonstandard or Jordanian deformation). We investigate the existence of quantum deformations that not only refer to the combined classical r-matrix and can be connected with the standard quantization by a twist, but such that all their algebraic elements (bialgebraic structure, universal $\mathcal{R}$-matrix and twisting element) have well defined limits. In section 2 we prove that this problem can be solved by constructing a quantum deformation $\mathcal{F}_{q\mathcal{J}}$ of the Jordanian twist $\mathcal{F}_{\mathcal{J}}$. This new quantum Jordanian twist $\mathcal{F}_{q\mathcal{J}}$ acts on the standard quantizations of the universal enveloping algebras and transforms them into the hybrid quantizations (we have borrowed this term from [9]). The twisting element $\mathcal{F}_{q\mathcal{J}}$ itself and all the corresponding twisted constructions have both natural limits. In section 3 we apply this twist to the quantum algebras based on $sl(2)$ and obtain the hybrid quantizations of $U(sl(2))$ and of the quantum affine algebra $U(\widehat{sl(2)})$,

$$\begin{aligned}U_q(sl(2)) &\xrightarrow{\mathcal{F}_{q\mathcal{J}}} U_{q\mathcal{J}}(sl(2))\\ U_q(\widehat{sl(2)}) &\xrightarrow{\mathcal{F}_{q\mathcal{J}}} U_{q\mathcal{J}}(\widehat{sl(2)}).\end{aligned}$$

The corresponding universal $\mathcal{R}$-matrices and their defining representations are also presented.

2. Quantum Jordanian twist for $U_q(B)$

Proposition 1. *The quantum Borel algebra $U_q(B)$*

$$[H, E] = E \qquad \begin{array}{l} \Delta(H) = H \otimes 1 + 1 \otimes H \\ \Delta(E) = E \otimes 1 + \mathrm{e}^{hH} \otimes E \end{array} \tag{2.1}$$

admits the twist with the element

$$\tilde{\mathcal{F}}_{q\mathcal{J}} = \mathrm{e}^{H\otimes\sigma} \qquad \sigma = \ln(E + \mathrm{e}^{hH}). \tag{2.2}$$

Proof. We shall demonstrate that the element (2.2) satisfies the factorized twist equations. The first of them is trivially fulfilled due to the primitivity of H. To check the second let us change the basis. The new generator

$$\breve{E} = E - 1 + \mathrm{e}^{hH} \tag{2.3}$$

has the same coproduct as E. Performing the substitution we obtain for $U_q(B)$ the relations

$$[H, \breve{E}] = \breve{E} + 1 - \mathrm{e}^{hH} \qquad \begin{array}{l} \Delta(H) = H \otimes 1 + 1 \otimes H \\ \Delta(\breve{E}) = \breve{E} \otimes 1 + \mathrm{e}^{hH} \otimes \breve{E} \end{array} \tag{2.4}$$

and σ from (2.2) becomes

$$\sigma = \ln(1 + \breve{E}).$$

The adjoint action of H on $\breve{E}$ differs from that of H on E by terms that are central. So, one obtains

$$\mathrm{e}^{\mathrm{ad}(H\otimes\sigma)} \circ \breve{E} \otimes 1 = \breve{E} \otimes \mathrm{e}^{\sigma} + (1 - \mathrm{e}^{hH}) \otimes (\mathrm{e}^{\sigma} - 1).$$

Now we can obtain the final form of the coproduct for e^{σ},

$$\begin{aligned} \tilde{\Delta}_{q\mathcal{J}}(\mathrm{e}^{\sigma}) &= \tilde{\Delta}_{q\mathcal{J}}(\breve{E} + 1) \\ &= \mathrm{e}^{\mathrm{ad}(H\otimes\sigma)} \circ (\breve{E} \otimes 1 + \mathrm{e}^{hH} \otimes \breve{E} + 1 \otimes 1) \\ &= \mathrm{e}^{\mathrm{ad}(H\otimes\sigma)} \circ (\breve{E} \otimes 1) + \mathrm{e}^{hH} \otimes \breve{E} + 1 \otimes 1 \\ &= \breve{E} \otimes \mathrm{e}^{\sigma} + 1 \otimes \mathrm{e}^{\sigma} = \mathrm{e}^{\sigma} \otimes \mathrm{e}^{\sigma}. \end{aligned} \tag{2.5}$$

Consequently, σ becomes primitive with respect to the deformed coproduct $\tilde{\Delta}_{q\mathcal{J}}$. Thus, the element $\tilde{\mathcal{F}}_{q\mathcal{J}}$ satisfies also the second of the factorized twist equations (1.4). This completes the proof. □

It will be useful to introduce now the twisting parameter ξ. This can be achieved by rescaling the generator $E \to \xi E$. So, proposition 1 is valid also for the parametric set of twisting elements

$$\tilde{\mathcal{F}}_{q\mathcal{J}}(h, \xi) = \mathrm{e}^{H\otimes\sigma} \qquad \sigma = \ln(\xi E + \mathrm{e}^{hH}). \tag{2.6}$$

Performing the twisting we obtain the smooth two-parameter set $\{\tilde{U}_{q\mathcal{J}}(B)(h, \xi)\}$ of Hopf algebras

$$[H, E] = E \qquad \begin{array}{l} \tilde{\Delta}_{q\mathcal{J}}(H) = H \otimes 1 + \mathrm{e}^{\mathrm{ad}(H\otimes\sigma)} \circ (1 \otimes H) \\ \tilde{\Delta}_{q\mathcal{J}}(E) = \dfrac{1}{\xi}(\mathrm{e}^{\sigma} \otimes \mathrm{e}^{\sigma} - \mathrm{e}^{\mathrm{ad}(H\otimes\sigma)} \circ (\mathrm{e}^{hH} \otimes \mathrm{e}^{hH})). \end{array} \tag{2.7}$$

This set describes the *hybrid quantization* that has the properties both of the standard quantization and of the Jordanian. For the limit values of the parameters we obtain the Hopf algebras whose characteristics are different from the generic ones. The boundary of the set corresponding to $h = 0$ gives the ordinary Jordanian quantization $\tilde{U}_{q\mathcal{J}}(B)(0, \xi) = U_{\mathcal{J}}(B)$.

The boundary $\tilde{U}_{q\mathcal{J}}(B)(h,0)$ describes the Hopf algebras that are equivalent to the standard deformation $U_q(B)$ but have the shifted coproduct for E. Such behaviour is in accordance with the limit form of the twisting element (2.6)

$$\tilde{\mathcal{F}}_{q\mathcal{J}}(h,0) = \mathrm{e}^{hH\otimes H}. \tag{2.8}$$

If we want to go back (when ξ tends to zero) to the initial algebra $U_q(B)$ the compensating Reshetikhin twist must be applied. The final construction is defined as follows.

Proposition 2. *The quantum Borel algebra* $U_q(B)$

$$[H,E] = E \qquad \begin{array}{l}\Delta(H) = H\otimes 1 + 1\otimes H\\ \Delta(E) = E\otimes 1 + \mathrm{e}^{hH}\otimes E\end{array}$$

admits the twist with the element

$$\mathcal{F}_{q\mathcal{J}}(h,\xi) = \mathrm{e}^{H\otimes\omega}\mathrm{e}^{-hH\otimes H} \qquad \omega = \ln((\xi E+1)\mathrm{e}^{hH}). \tag{2.9}$$

Proof. The first factor of the twisting element (2.9) transforms the quantized Borel $U_q(B)$ into its dual Hopf algebra:

$$[H,E] = E \qquad \begin{array}{l}\Delta_{qR}(H) = H\otimes 1 + 1\otimes H\\ \Delta_{qR}(E) = E\otimes \mathrm{e}^{-hH} + 1\otimes E.\end{array} \tag{2.10}$$

The substitution $E \to E\mathrm{e}^{-hH}$ brings us back to the initial Borel in the form (2.1) and also changes ω in (2.9) for σ (as defined in (2.6)). After this we find ourselves in the situation of the proposition 1. □

Applying the twist (2.9) to $U_q(B)$ we obtain the Hopf algebras with the following defining relations:

$$[H,E] = E \qquad \begin{array}{l}\Delta_{q\mathcal{J}}(H) = H\otimes 1 + \mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes H)\\ \Delta_{q\mathcal{J}}(E) = \frac{1}{\xi}((\mathrm{e}^{\omega}\otimes\mathrm{e}^{\omega}) - \mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(\mathrm{e}^{-hH}\otimes\mathrm{e}^{-hH}) - 1\otimes 1).\end{array} \tag{2.11}$$

We have two correlated smooth sets: the set $\{U_{q\mathcal{J}}(B)(h,\xi)\}$ of hybrid quantizations (2.11) and the set of quantum Jordanian twists $\{\mathcal{F}_{q\mathcal{J}}(h,\xi)\}$. For each point $U_{q\mathcal{J}}(B)(h,\xi)$ of the first set there exits a twist $\mathcal{F}_{q\mathcal{J}}^{-1}(h,\xi)$ that connects this point with the algebra $U_{q\mathcal{J}}(B)(h,0)$. Now the sets have the appropriate boundary behaviour:

$$\begin{array}{ccc} & U_{q\mathcal{J}}(B)(h,\xi) & \\ & {\scriptstyle h\to 0}\swarrow \qquad \searrow{\scriptstyle \xi\to 0} & \\ U_{\mathcal{J}}(B) = U_{q\mathcal{J}}(B)(0,\xi) & & U_{q\mathcal{J}}(B)(h,0) = U_q(B)\end{array} \tag{2.12}$$

$$\begin{array}{ccc} & \mathcal{F}_{q\mathcal{J}}(h,\xi) & \\ & {\scriptstyle h\to 0}\swarrow \qquad \searrow{\scriptstyle \xi\to 0} & \\ \mathcal{F}_{\mathcal{J}}^{c} = \mathcal{F}_{q\mathcal{J}}(0,\xi) & & \mathcal{F}_{q\mathcal{J}}(h,0) = 1\otimes 1.\end{array} \tag{2.13}$$

For the internal points of $\{U_{q\mathcal{J}}(B)(h,\xi)\}$ the defining relations (2.11) can be written in a compact form

$$[H,\mathrm{e}^{\omega}] = \mathrm{e}^{\omega} - \mathrm{e}^{hH} \qquad \begin{array}{l}\Delta_{q\mathcal{J}}(H) = H\otimes 1 + \mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes H)\\ \Delta_{q\mathcal{J}}(\mathrm{e}^{\omega}) = \mathrm{e}^{\omega}\otimes\mathrm{e}^{\omega}.\end{array} \tag{2.14}$$

However, such a description becomes incomplete on the boundary $\{U_{q\mathcal{J}}(B)(h,0)\}$ where $\omega(h,0) = hH$.

3. Jordanian deformations of quantum algebras

3.1. Hybrid quantization $U_{q\mathcal{J}}(sl(2))$

The quantum Jordanian twists $\mathcal{F}_{q\mathcal{J}}(h,\xi)$ and $\tilde{\mathcal{F}}_{q\mathcal{J}}(h,\xi)$ can be applied to any Hopf algebra containing the quantized Borel algebra (2.1). Let us start with the standard quantization $U_q(sl(2))$:

$$\begin{array}{ll} [H,E_\pm]=\pm E_\pm & \Delta_q(H)=H\otimes 1+1\otimes H \\ [E_+,E_-]=\dfrac{\mathrm{e}^{hH}-\mathrm{e}^{-hH}}{1-\mathrm{e}^{-h}} & \Delta_q(E_+)=E_+\otimes 1+\mathrm{e}^{hH}\otimes E_+ \\ & \Delta_q(E_-)=E_-\otimes \mathrm{e}^{-hH}+1\otimes E_-. \end{array} \tag{3.1}$$

Applying the twist $\mathcal{F}_{q\mathcal{J}}(h,\xi)$ in the form given in (2.9) we obtain the two-parameter set $\{U_{q\mathcal{J}}(sl(2))(h,\xi)\}$ of quantum deformations that are the *hybrids* of standard and Jordanian ones:

$$\begin{array}{ll} [H,E_\pm]=\pm E_\pm & \Delta_{q\mathcal{J}}(H)=H\otimes 1+\mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes H) \\ [E_+,E_-]=\dfrac{\mathrm{e}^{hH}-\mathrm{e}^{-hH}}{1-\mathrm{e}^{-h}} & \Delta_{q\mathcal{J}}(E_+)=\frac{1}{\xi}(\mathrm{e}^{\omega}\otimes\mathrm{e}^{\omega}-\mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(\mathrm{e}^{-hH}\otimes\mathrm{e}^{-hH})-1\otimes 1) \\ & \Delta_{q\mathcal{J}}(E_-)=E_-\otimes\mathrm{e}^{-\omega}+\mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes E_-). \end{array} \tag{3.2}$$

The Hopf algebras $U_q(sl(2))$ and $U_{\mathcal{J}}(sl(2))$ form the boundaries of this set:

$$\begin{array}{ccc} & U_{q\mathcal{J}}(B)(h,\xi) & \\ & {}_{h\to 0}\swarrow \qquad \searrow_{\xi\to 0} & \\ U_{\mathcal{J}}(sl(2))=U_{q\mathcal{J}}(sl(2))(0,\xi) & & U_{q\mathcal{J}}(sl(2))(h,0)=U_q(sl(2)). \end{array} \tag{3.3}$$

When the internal subset $\{U_{q\mathcal{J}}(sl(2))(h,\xi)|h>0,\xi>0\}$ is considered, the compact form of the defining relations can be used,

$$\begin{array}{ll} [H,\mathrm{e}^{\omega}]=\mathrm{e}^{\omega}-\mathrm{e}^{hH} & \Delta_{q\mathcal{J}}(H)=H\otimes 1+\mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes H) \\ [H,E_-]=-E_- & \Delta_{q\mathcal{J}}(\mathrm{e}^{\omega})=\mathrm{e}^{\omega}\otimes\mathrm{e}^{\omega} \\ [E_-,\mathrm{e}^{\omega}]_{\mathrm{e}^h}=\xi\dfrac{1-\mathrm{e}^{2hH}}{1-\mathrm{e}^{-h}} & \Delta_{q\mathcal{J}}(E_-)=E_-\otimes\mathrm{e}^{-\omega}+\mathrm{e}^{\mathrm{ad}(H\otimes\omega)}\circ(1\otimes E_-). \end{array} \tag{3.4}$$

The algebra (3.1) is quasitriangular with the universal $\mathcal{R}$-matrix

$$\mathcal{R}_q=\mathrm{e}^{hH\otimes H}\sum_{n=0}^{\inf}\frac{(1-\mathrm{e}^{-h})^n}{[n]!}(E_-\otimes E_+)^n\mathrm{e}^{\frac{1}{4}hn(n-1)} \qquad [n]=\frac{\mathrm{e}^{\frac{nh}{2}}-\mathrm{e}^{\frac{-nh}{2}}}{\mathrm{e}^{\frac{h}{2}}-\mathrm{e}^{\frac{-h}{2}}}. \tag{3.5}$$

The same is true for the hybrid algebra $U_{q\mathcal{J}}(sl(2))(h,\xi)$. According to the general properties of twisted quasitriangular algebras (see equation (1.5)) $U_{q\mathcal{J}}(sl(2))(h,\xi)$ has the following $\mathcal{R}$-matrix:

$$\mathcal{R}_{q\mathcal{J}}=\mathrm{e}^{\omega\otimes H}\mathrm{e}^{-hH\otimes H}\mathcal{R}_q\mathrm{e}^{hH\otimes H}\mathrm{e}^{-H\otimes\omega}. \tag{3.6}$$

In a case of the smooth set of quantized algebras the classical limit depends on how we fix the linear subvariety that describes the deformation quantization. If we want to disclose the hybrid properties of the set $\{U_{q\mathcal{J}}(sl(2))(h,\xi)\}$ we are to find a smooth curve intermediate between the standard deformation subvariety $\{U_{q\mathcal{J}}(sl(2))(h,0)|h\geqslant 0\}$ and the pure twist subvariety $\{U_{q\mathcal{J}}(sl(2))(0,\xi)|\xi\geqslant 0\}$. Obviously, it is sufficient to consider a linear subvariety $\{U_{q\mathcal{J}}(sl(2))(\zeta\xi,\xi)|\xi\geqslant 0,\zeta>0\}$ where we had put $h=\zeta\xi$. In the corresponding set $\{\mathcal{R}_{q\mathcal{J}}|h=\zeta\xi\}$ of hybrid $\mathcal{R}$-matrices (3.6) we let ξ be in the neighbourhood of zero and extract the classical r-matrix

$$r_{q\mathcal{J}}=E_+\wedge H+\zeta(H\otimes H+E_-\otimes E_+). \tag{3.7}$$

This expression is the well known hybrid solution [4] of the classical Yang–Baxter equation.

3.2. Hybrid quantum affine algebra $U_{q\mathcal{J}}(\widehat{sl(2)})$

The explicit construction of Jordanian twist [3], extended Jordanian twist [11] and chains of twists [12] provided the possibility to obtain the mixed quantizations for current algebras—the twisted Yangians [13–15]. Analogously, with the help of the q-Jordanian twist $\mathcal{F}_{q\mathcal{J}}(h, \xi)$ we can obtain the hybrid quantizations for Kac–Moody algebras.

Let us consider, for example, the quantum affine algebra $U_q(\widehat{sl(2)})$ [16, 17] defined as a deformed infinite-dimensional Lie algebra with the Cartan matrix

$$A = (a_{ij}) = [(\lambda_i, \lambda_j)] = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \qquad i, j = 0, 1$$

the generators H_i, $E_{\pm\lambda_i}$, D and the relations

$$\begin{aligned}
&[H_i, E_{\pm\lambda_j}] = \pm\tfrac{1}{2} a_{ij} E_{\pm\lambda_j} \\
&[E_{\lambda_i}, E_{-\lambda_j}] = \delta_{ij} \frac{\mathrm{e}^{hH_i} - \mathrm{e}^{-hH_i}}{1 - \mathrm{e}^{-h}} \\
&[D, E_{\pm\lambda_i}] = \pm\delta_{i0} E_{\pm\lambda_i} \qquad i, j = 0, 1 \\
&[H_i, H_j] = [H_i, D] = 0 \\
&(\mathrm{ad}_q E_{\pm\lambda_i})^{1-a_{ij}} \circ E_{\pm\lambda_j} = 0 \qquad i \neq j.
\end{aligned} \tag{3.8}$$

Here ad_q is the q-adjoint operator

$$\mathrm{ad}_q E_{\lambda_i} \circ E_{\lambda_j} = E_{\lambda_i} E_{\lambda_j} - \mathrm{e}^{h(\lambda_i, \lambda_j)} E_{\lambda_j} E_{\lambda_i}.$$

We shall put $q = \mathrm{e}^{\frac{1}{2}h}$ and introduce the rescaled generators

$$e_{\pm\lambda_i} = \mathrm{e}^{-\frac{1}{4}h} E_{\lambda_i}. \tag{3.9}$$

Let $\delta = \lambda_0 + \lambda_1$ be the minimal imaginary root of $\widehat{sl(2)}$. Then, the so-called normal ordering [18] in the system of positive roots Λ_+ is fixed as follows:

$$\lambda_0, \lambda_0 + \delta, \ldots, \lambda_0 + l\delta, \ldots, \delta, 2\delta, \ldots, \ldots, \lambda_1 + (n+1)\delta, \lambda_1 + n\delta, \ldots, \lambda_1. \tag{3.10}$$

According to this ordering the generators for composite roots are obtained as follows:

$$\begin{aligned}
&e'_\delta = [2]^{-1}[e_{\lambda_0}, e_{\lambda_1}]_q \qquad e'_{\lambda_0+n\delta} = (-1)^n (\mathrm{ad}\, e'_\delta)^n \circ e_{\lambda_0} \\
&e'_{\lambda_1+n\delta} = (\mathrm{ad}\, e'_\delta)^n \circ e_{\lambda_1} \qquad e'_{n\delta} = [2]^{-1}[e_{\lambda_0+(n-1)\delta}, e_{\lambda_1}]_q.
\end{aligned}$$

(The q-numbers above are the same as in (3.5).) Finally, the generators $e_{n\delta}$ are defined by means of the Schur polynomials:

$$e'_{n\delta} = \sum_{p_1+2p_2+\cdots+np_n=n} \frac{(q^2 - q^{-2})^{\sum p_i - 1}}{p_1! \cdots p_n!} e_\delta^{p_1} e_{2\delta}^{p_2} \cdots e_{n\delta}^{p_n}.$$

The generators for the negative roots are defined with the help of the involution

$$(H_i)^* = -H_i \qquad (e_{\pm\lambda_i})^* = e_{\mp\lambda_i} \qquad h^* = -h.$$

In terms of these generators the universal $\mathcal{R}$-matrix of $U_q(\widehat{sl(2)})$ has the form [19]

$$\begin{aligned}
\mathcal{R}^{DJ} = &\left(\overrightarrow{\prod_{n\leqslant 0}} \exp_q((q - q^{-1}) e_{\alpha+n\delta} \otimes e_{-\alpha-n\delta}) \right) \exp\left(\sum_{n>0} \frac{n(e_{n\delta} \otimes e_{-n\delta})}{q^{2n} - q^{-2n}} \right) \\
&\times \left(\overleftarrow{\prod_{n\leqslant 0}} \exp_q((q - q^{-1}) e_{\beta+n\delta} \otimes e_{-\beta-n\delta}) \right) \mathcal{K}
\end{aligned} \tag{3.11}$$

where $\mathcal{K}$ represents

$$\mathcal{K} = \exp\left(\sum_{i,j} 2hd_{ij} H_i \otimes H_j\right)$$

d is the inverse of the extended (nondegenerate) Cartan matrix $\tilde{a}$ [20] and the q-exponent is defined as the series

$$\exp_q \equiv \sum \frac{x^n}{(n)_{q^{-2}}} \qquad (n)_{q^{-2}} = \frac{q^{-2n}-1}{q^{-2}-1}.$$

Note that the order of q-exponents in (3.11) is direct in the first product ($\rightarrow$) and inverse in the second one ($\leftarrow$).

Any quantum Borel subalgebra $U_q(B) \in U_q(\widehat{sl(2)})$ can be used as a carrier algebra to perform the quantum Jordanian twisting. If we have to consider representations of the corresponding quasitriangular quantum algebras the simplest choice is to take the Hopf subalgebra generated by H_0 and E_{λ_0}. The twist deformation is performed by the element (see (2.9))

$$\mathcal{F}_{q\mathcal{J}}(h,\xi) = \mathrm{e}^{H_0\otimes\omega_0}\mathrm{e}^{-hH_0\otimes H_0} \qquad \omega_0 = \ln((\xi E_{\lambda_0}+1)\mathrm{e}^{hH_0}) \tag{3.12}$$

and produces the Jordanian quantum affine algebra $U_{q\mathcal{J}}(\widehat{sl(2)})$. It has the commutators defined by (3.8) and the deformed coproducts:

$$\begin{aligned}
&\Delta_{q\mathcal{J}}(H_i) = H_i\otimes 1 + \mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(1\otimes H_i)\\
&\Delta_{q\mathcal{J}}(D) = D\otimes 1 + \mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(1\otimes D)\\
&\Delta_{q\mathcal{J}}(E_{\lambda_0}) = \frac{1}{\xi}(\mathrm{e}^{w}\otimes\mathrm{e}^{w} - \mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(\mathrm{e}^{-hH_0}\otimes\mathrm{e}^{-hH_0}) - 1\otimes 1)\\
&\Delta_{q\mathcal{J}}(E_{-\lambda_0}) = E_{-\lambda_0}\otimes\mathrm{e}^{-w} + \mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(1\otimes E_{-\lambda_0})\\
&\Delta_{q\mathcal{J}}(E_{\lambda_1}) = (E_{\lambda_1}\otimes\mathrm{e}^{-w})(\mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(1\otimes\mathrm{e}^{hH_0})) + \mathrm{e}^{h(H_1+H_0)}\otimes E_{\lambda_1}\\
&\Delta_{q\mathcal{J}}(E_{-\lambda_1}) = (E_{-\lambda_1}\otimes\mathrm{e}^{+w})(\mathrm{e}^{\mathrm{ad}(H_0\otimes w)}\circ(1\otimes\mathrm{e}^{-h(H_1+H_0)})) + \mathrm{e}^{-hH_0}\otimes E_{-\lambda_1}.
\end{aligned} \tag{3.13}$$

The twisted (hybrid) universal $\mathcal{R}$-matrix for $U_{q\mathcal{J}}(\widehat{sl(2)})$ has the following form:

$$\begin{aligned}
\mathcal{R}^{DJ}_{q\mathcal{J}} &= \mathrm{e}^{\omega\otimes H}\mathrm{e}^{-hH\otimes H}\left(\overrightarrow{\prod_{n\leqslant 0}}\exp_q((q-q^{-1})\boldsymbol{e}_{\alpha+n\delta}\otimes\boldsymbol{e}_{-\alpha-n\delta})\right)\\
&\quad\times\exp\left(\sum_{n>0}\frac{n(\boldsymbol{e}_{n\delta}\otimes\boldsymbol{e}_{-n\delta})}{q^{2n}-q^{-2n}}\right)\left(\overleftarrow{\prod_{n\leqslant 0}}\exp_q((q-q^{-1})\boldsymbol{e}_{\beta+n\delta}\otimes\boldsymbol{e}_{-\beta-n\delta})\right)\\
&\quad\times\mathcal{K}\mathrm{e}^{hH\otimes H}\mathrm{e}^{-H\otimes\omega}.
\end{aligned} \tag{3.14}$$

It satisfies the parametric QYBE

$$\mathcal{R}_{12}(z_1/z_2)\mathcal{R}_{13}(z_1/z_3)\mathcal{R}_{23}(z_2/z_3) = \mathcal{R}_{23}(z_2/z_3)\mathcal{R}_{13}(z_1/z_3)\mathcal{R}_{12}(z_1/z_2).$$

In the fundamental representation of $sl(2)$ we obtain the hybrid matrix solution:

$$d(\mathcal{R}^{DJ}_{q\mathcal{J}}) = \frac{1-z}{g^{\frac{3}{2}}(1-zq^{-2})}\exp\left(\sum_{n>0}\frac{z^n}{n}\frac{q^n-q^{-n}}{q^n+q^{-n}}\right)\begin{pmatrix} a_1 & sq & -s & s^2\\ 0 & q & a_2 & s\\ 0 & za_2 & q & -sq\\ 0 & 0 & 0 & a_1\end{pmatrix} \tag{3.15}$$

where

$$a_1 = \frac{q^2-z}{1-z} \qquad a_2 = \frac{q^2-1}{1-z} \qquad s = \frac{\xi}{1+q}.$$

The expression for the universal $\mathcal{R}$-matrix (3.14) as well as for its defining representation (3.15) describes a smooth variety of solutions of QYBE. This is the two-dimensional variety with the coordinates h and ξ and with the spectral parameter z:

$$\begin{array}{ccc} & \mathcal{R}^{DJ}_{q\mathcal{J}}(h,\xi) & \\ {}^{h\to 0}\swarrow & & \searrow{}_{\xi\to 0} \\ \mathcal{R}_{\mathcal{J}} = \mathcal{R}^{DJ}_{q\mathcal{J}}(0,\xi) & & \mathcal{R}^{DJ}_{q\mathcal{J}}(h,0) = \mathcal{R}^{DJ}. \end{array} \tag{3.16}$$

When ξ goes to zero we return to the initial quantum affine algebra $U_q(\widehat{sl(2)})$ and the corresponding $\mathcal{R}$-matrix (3.11). In the limit $h \to 0$ we obtain the nonstandard quantization $U_{\mathcal{J}}(\widehat{sl(2)})$ of the affine algebra $U(\widehat{sl(2)})$ performed by the Jordanian twist

$$\mathcal{F}_J = \mathrm{e}^{H_0\otimes\sigma_0}$$

with $\sigma_0 = \ln(1 + E_{\lambda_0})$. The $\mathcal{R}$-matrix in this limit case becomes the ordinary Jordanian.

4. Conclusions

We have demonstrated that there exists the hybrid quantization $U_{q\mathcal{J}}(sl(2))$ with well defined natural limits with respect to the two deformation parameters h and ξ. Moreover, each quantum algebra $U_{q\mathcal{J}}(sl(2))$ can be considered together with the twisting element $\mathcal{F}_{q\mathcal{J}}$, that connects it with the corresponding standard quantization $U_q(sl(2))$. Both natural limits exist also for the triples $(U_{q\mathcal{J}}(sl(2)), \mathcal{F}_{q\mathcal{J}}, \mathcal{R}^{DJ}_{q\mathcal{J}})$. Such limit behaviour illustrates the difference between previously obtained combined quantizations [6–8, 10] and the deformation $U_{q\mathcal{J}}$ produced by the quantum Jordanian twist. In contrast to the cases of (q,ξ)-deformation by Ballesteros *et al* [6] (BHP deformation), the constructions proposed by Abdesselam *et al* [7, 8] and Stolin [10] in the triple $(U_{q\mathcal{J}}(sl(2)), \mathcal{F}_{q\mathcal{J}}, \mathcal{R}^{DJ}_{q\mathcal{J}})$ there are no singularities when $q \to 1$. All the above-mentioned quantum algebras have the hybrid classical r-matrix of the type (3.7) or similar to it. It must be stressed that a classical r-matrix may refer to different quantizations. This is just the case we have here. Consider, in particular, the BHP r-matrix. It depends on two parameters and in the two limit cases becomes equal to $r_{\mathcal{DJ}}$ (Drinfeld–Jimbo) and $r_{\mathcal{J}}$ (Jordanian), correspondingly. Nevertheless, the Hopf algebra of the BHP deformation has no Jordanian limit since its structure constants diverge when $q \to 1$ (see formulae (4.5) and (4.6) of [6] and the comments at the end of section 4.1.1 therein). The same is true for the coalgebra preserving map (constructed in the section 3.2 of [9]) that realizes the equivalence of the BHP and the standard deformations: it also has no limit for $q \to 1$. Thus, it is clear that the quantum Jordanian twist cannot be considered as a 'twisted version' of the equivalence transformation proposed in [9].

The BHP deformation and the q-Jordanian twist produce two-dimensional varieties of Hopf algebras, which have the boundary in the orbit of the standard quantum algebra, but they are nonequivalent. The variety $\{U_{q\mathcal{J}}(sl(2))\}$ has additionally the second boundary in the orbit of the Jordanian quantization (both boundaries intersect in the orbit of the classical algebra $U(sl(2))$). This means that the set $\{U_{q\mathcal{J}}(sl(2))\}$ and the (q,ξ)-deformations represent different 'sheets' of quantum algebras.

In [9] the BHP deformation was considered not to be an authentic hybrid quantization. The reason was the equivalence of the standard and the BHP deformations mentioned above. From our point of view, the requirement of nonequivalence to U_q is too strong in the context of hybrid quantizations. The main criterion must be the possibility for a Hopf algebra to be submerged in a two-dimensional smooth variety whose boundaries are U_q and $U_{\mathcal{J}}$. As shown in [21] this implies that it is a quantization of a hybrid classical r-matrix (3.7).

Notice that the Jordanian quantum affine algebra $U_{q\mathcal{J}}(\widehat{sl(2)})$ constructed in section 3 is an example of *twisted quantum nontwisted affine algebras*, $U_{q\mathcal{J}}(\widehat{sl(2)}) = U_{q\mathcal{J}}(A_1^{(1)})$. The word 'twisted' in the term 'twisted affine algebra' (introduced by Kac [20]) has a meaning different from that of the Drinfeld deformation procedure [1]. This is why the term 'Jordanian' is preferable here.

The quantum Jordanian twist $\mathcal{F}_{q\mathcal{J}}$ can be applied to any Hopf algebra containing the quantum Borel subalgebra $U_q(B)$. In particular, it can be used to produce hybrid deformations of the twisted affine algebras (the Kac–Moody algebras listed in tables Aff2 and Aff3 in [20]).

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